Orbits, Symplectic Structures and Representation Theory

Bertram Kostant

We introduce a general approach to unitary representations for all Lie groups. An underlying feature is a study of symplectic manifolds $X^{2n}$ (i.e., there exists a closed non-singular 2-form on $X$). If $[\omega] \in H^2(X, \mathbb{R})$ is an integral class there is an associated affinely connected Hermitian line bundle $L$ over $X$ which is unique if $X$ is simply connected. Given a complex involutory totally singular distribution $F^*$ on $X$ there is an associated cohomology theory $H(L_F)$ for the sheaf $L_F$ of local sections of $L$ which are constant along $F$. There then exists a Lie subalgebra $\mathfrak{a}$ of the Lie algebra (under Poisson bracket) $\mathfrak{c}$ of all smooth functions on $X$ which naturally operates on $H(L_F)$. For an element of $\mathfrak{c}$ to operate on $H(L_F)$ in the case when $X$ is a cotangent space is the inverse operation of taking the symbol of a differential operator. In general it is analogous to what the physicists call quantizing a function.

Next one gives a complete classification of all symplectic homogeneous spaces for all Lie groups. They are related to orbits in the dual of the Lie algebra. The correspondence with an orbit maps the Lie algebra of the group into $\mathfrak{a}$ (see above) and yelds a representation of the group on $H(L_F)$. The theory thus obtained embraces the Borel-Weil theory for compact groups, the Kirilov theory for nilpotent groups and Harish-Chandra-Gelfand theory for semi-simple groups.

Massachusetts Institute of Technology